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The hidden phase of Fock states; quantum non-local effects

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Abstract. We revisit the question of how a definite phase between Bose-Einstein condensates can spontaneously appear under the effect of measurements. We first consider a system that is the juxtaposition of two subsystems in Fock states with high populations, and assume that successive individual position measurements are performed. Initially, the relative phase is totally undefined, and no interference effect takes place in the first position measurement. But, while successive measurements are accumulated, the relative phase becomes better and better defined, and a clear interference pattern emerges. It turns out that all observed results can be interpreted in terms of a pre-existing, but totally unknown, relative phase, which remains exactly constant during the experiment. We then generalize the results to more condensates. We also consider other initial quantum states than pure Fock states, and distinguish between intrinsic phase of a quantum state and phase induced by measurements. Finally, we examine the case of multiple condensates of spin states. We discuss a curious quantum effect, where the measurement of the spin angular momentum of a small number of particles can induce a big angular momentum in a much larger assembly of particles, even at an arbitrary distance. This spin observable can be macroscopic, analogous to the pointer of a measurement apparatus, which illustrates the non-locality of standard quantum mechanics with particular clarity. The effect can be described as the teleportation at arbitrary distances of the continuous classical result of a local experiment. The EPR argument, transposed to this case, takes a particularly convincing form since it does not involve incompatible measurements and deals only with macroscopic variables.

PACS. 03.65.Ta Foundations of quantum mechanics; measurement theory – 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.75.Gg Entanglement and decoherence in Bose-Einstein condensates

It is well-known in quantum mechanics that one cannot construct a quantum state where the number of particles and the phase are both arbitrarily well defined; they are actually incompatible observables, related by a Heisenberg type uncertainty relation, as position and momentum of a single particle. For a general discussion of the phase operator in quantum mechanics, see for instance references [1–3]. An usual illustration of the relation between phase and particle number is given by the so-called "coherent states" or "Schrödinger semi-classical states", where fluctuations of the number of particles are used in order to define a phase. These states are often discussed in the context of electromagnetism [4], but they also apply to Bose-Einstein condensates, on which we focus the interest here — more precisely, multiple condensates and their relative phase.

For instance we consider a physical system that is the juxtaposition of two systems in Fock states (number states), described by the state vector:

$$|\Phi_0\rangle = |N_a : \varphi_a \ ; \ N_b : \varphi_b\rangle \qquad (1)$$

where N_a particles are condensed in the same single particle quantum state φ_a, and N_b particles in the same quantum state φ_b. These two states may correspond to plane waves with momenta $\mathbf{k}_a$ and $\mathbf{k}_b$, a case in which one can expect the occurrence of an interference pattern with spatial frequency $\mathbf{k}_a - \mathbf{k}_b$. Nevertheless, the position of this pattern depends of the relative phase of the waves, which is completely unknown in a state such as (1). One could then wonder whether an interference effect is observable at all under these conditions.

The question was studied theoretically by several authors [5–9]; see also references [10–13]. The result is that an interference pattern is in fact observed, with a phase that spontaneously emerges from the quantum measurement process itself. The detection of the first particle occurs at a completely random position, but this position provides a first information on the value of the phase; the second particle is then detected at a position that is correlated with the first, and this measurements makes the information on the phase more precise; as more and more detection events are accumulated, the phase becomes better and better defined. In practice, a relatively small number of detections is sufficient to determine the relative

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phase with good accuracy. Nevertheless, before the first measurement, according to standard quantum mechanics there is no way whatsoever to predict the value that this phase will spontaneously choose. In other words, repeating the same experiment from the same initial conditions (1) will lead to another, completely independent, value of the relative phase. The spontaneous appearance of a relative phase under the effect of measurement is related to the notion of spontaneous symmetry breaking of the conservation of the number of particles, and to non-zero average values of the field operator in superfluid systems [14,15]. It also relates to Anderson's intriguing question, as quoted by Leggett [16]: "Do superfluids which have never seen each other possess a definite relative phase?" [17,18].

In this article, we revisit this question by using a slightly different method from that of references [5–11,13]; we propose a straightforward analytical calculation that does not require the introduction of any (incoherent) fluctuation of the particle numbers, as in [7,8] for instance, but deals directly with pure Fock states; it reproduces the stochastic simulations of [5] with only one approximation (total number of particles much larger than the number of measurements). In Section 1 we discuss how the process of measurement, applied to a pair of systems that are initially in Fock states, therefore with no well defined phase, can make such a phase appear spontaneously; in Section 2 we apply our calculations to more general cases and discuss how an intrinsic phase, contained in the initial state, can combine with a spontaneous phase induced by measurements; in Section 3, we discuss multiple condensates in the context of spin states and show in particular how the measurement of a small number of spins may induce an angular momentum for a much larger assembly of spins, a curious non-local quantum effect.

1 Spontaneous phase induced by measurements on Fock states

We first recall a general result of quantum mechanics. We consider a system in a quantum state $|\Psi_0\rangle$, and a series of observables:

$$\begin{aligned} &A \text{ with eigenvalues } a_i;\ P(a_i) \text{ is the projector onto the corresponding eigenstate(s)} \\ &B \text{ with eigenvalues } b_j;\ P(b_j) \text{ is the projector onto the corresponding eigenstate(s)} \\ &C \quad \text{etc.} \end{aligned} \tag{2}$$

If we assume that all these observables commute with each other, then the probability of observing in a combined measurement the results a_i, b_j, c_k, etc. can simply be written:

$$\langle \Phi_0 | P(a_i)\ P(b_j)\ P(c_k) \ldots | \Phi_0 \rangle \tag{3}$$

(the usual intrication of projectors in the so called Wigner formula is not necessary here because the operators commute). Whether or not the wave packet reduction postulate is applied does no matter; in other words, all measurements can be made simultaneously, or one after the other, and the order of measurements is irrelevant. What is assumed, nevertheless, is that the sequence of measurements covers a time that is negligible in comparison with the time constants associated with the intrinsic evolution of the physical system; this is why no evolution operator has to be introduced in the formula. We now apply (3) to position measurements inside the overlap regions of two or more condensates.

1.1 A simple case: two highly populated states

We begin with a simple case, where the system is just a juxtaposition of two Bose-Einstein condensates in plane waves:

$$|\Phi_0\rangle = |N_a : \mathbf{k}_a\ ;\ N_b : \mathbf{k}_b\rangle. \tag{4}$$

Here $\mathbf{k}_a$ and $\mathbf{k}_b$ are single particle states of well defined momentum normalized in a box with periodic boundary conditions; we assume that N_a and N_b are large numbers. The probability for detecting a particle at point $\mathbf{r}$ corresponds to the following operator, similar to the projectors $P(a_i)$ introduced above:

$$\Psi^\dagger(\mathbf{r})\ \Psi(\mathbf{r}) = \sum_{i=1}^{N} |i : \mathbf{r}\rangle\langle i : \mathbf{r}|. \tag{5}$$

Here $\Psi(\mathbf{r})$ is the usual field operator, and $|i : \mathbf{r}\rangle$ is the one particle state corresponding to a perfect localization of particle i at point $\mathbf{r}$; $N = N_a + N_b$ is the total number of particles. To make the operator $\Psi^\dagger(\mathbf{r})\ \Psi(\mathbf{r})$ really similar to a projector with eigenvalues 0 and 1, one has to integrate it over some small $\mathbf{r}$ domain $\Delta_\mathbf{r}$ centered around $\mathbf{r}$; assuming that $\Delta_\mathbf{r}$ is sufficiently small ensures that the probability of finding two particles or more in $\Delta_\mathbf{r}$ is negligible. For the sake of simplicity we do not write these integrations explicitly; in other words we write probability densities instead of probabilities, but it would be easy to come back to real probabilities by multiplying by an appropriate power of $\Delta_\mathbf{r}$. The field operator can be expanded onto the annihilation operators $a_\mathbf{k}$ of momentum states according to:

$$\Psi(\mathbf{r}) \sim \sum_{\mathbf{k}}\ e^{i\mathbf{k}\cdot\mathbf{r}}\ a_\mathbf{k} \tag{6}$$

where, as usual, the sum over $\mathbf{k}$ ranges over the values allowed by periodic boundary conditions in a box.

We now calculate the probability for observing one particle at position $\mathbf{r}_1$, another at position $\mathbf{r}_2$, still another at position $\mathbf{r}_3$, ... one at position $\mathbf{r}_P$. We consider that all positions $\mathbf{r}_1$, $\mathbf{r}_2$, ... are different, so that all operators Ψ and $\Psi^\dagger$ commute (more precisely, we assume that the small integration domains $\Delta_{\mathbf{r}_1}$, $\Delta_{\mathbf{r}_{21}}$, ... do not overlap). This allows us to write this probability as:

$$\langle \Phi_0 | \Psi^\dagger(\mathbf{r}_1)\Psi^\dagger(\mathbf{r}_2)\Psi^\dagger(\mathbf{r}_3)\ldots\Psi(\mathbf{r}_3)\Psi(\mathbf{r}_2)\Psi(\mathbf{r}_1)|\Phi_0\rangle \tag{7}$$

or, if (6) is used:

$$\sim \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_1'} e^{i\left(\mathbf{k}_1-\mathbf{k}_1'\right)\cdot\mathbf{r}_1} \sum_{\mathbf{k}_2} \sum_{\mathbf{k}_2'} e^{i\left(\mathbf{k}_2-\mathbf{k}_2'\right)\cdot\mathbf{r}_2} \times \langle \Phi_0 | a^\dagger_{\mathbf{k}_1'}\ a^\dagger_{\mathbf{k}_2'} \ldots a^\dagger_{\mathbf{k}_P'}\ a_{\mathbf{k}_P} \ldots a_{\mathbf{k}_2} a_{\mathbf{k}_1} | \Phi_0 \rangle. \quad (8)$$

Acting on $|\Phi_0\rangle$ given by (4), the annihilation operators a's in the second line of this formula give zero unless the values of the $\mathbf{k}$ and $\mathbf{k}'$ momenta are all equal to $\mathbf{k}_a$ or $\mathbf{k}_b$. In addition, the series of $\mathbf{k}$'s must contain the same number of $\mathbf{k}_a$'s (and $\mathbf{k}_b$'s) as the series of $\mathbf{k}'$'s; otherwise, the product of a and $a\dagger$ operators creates a ket orthogonal to $|\Phi_0\rangle$, and the second line vanishes. When this condition is met, each $a_{\mathbf{k}_{a,b}}$ or $a^\dagger_{\mathbf{k}_{a,b}}$ introduces a factor $\sqrt{N_{a,b} \pm n}$, where n is some integer ranging from 0 and P (the number of measurements).

At this point we make our only approximation: we assume that P is much smaller than N_a and N_b, so that we can replace all these factors by $\sqrt{N_{a,b}}$. Now, in the first line of (8), consider for instance the two first summations over $\mathbf{k}_1$ and $\mathbf{k}_1'$; if one of these vectors is chosen as $\mathbf{k}_a$, the other as $\mathbf{k}_b$, one obtains a contribution $\sqrt{N_a N_b} e^{\pm i(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}_1}$; if both are equal to the same value, $\mathbf{k}_a$ or $\mathbf{k}_b$, one gets a contribution $N_a + N_b = N$. The same result is obviously also valid for all other sums. Therefore, if we introduce the notation:

$$\begin{cases} F_0(\mathbf{r}) = N_a + N_b = N \\ F_{\pm 1}(\mathbf{r}) = \sqrt{N_a N_b} e^{\pm i(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}} = xN\ e^{\pm i(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}} \end{cases} \quad (9)$$

where x is the contrast ratio x ($0 \le x \le 1$) defined by:

$$N = N_a + N_b; \qquad x = \frac{2\sqrt{N_a N_b}}{N} \quad (10)$$

we can rewrite (8) as:

$$\sim \sum_{\{\sum\ q_i=0\}} F_{q_1}(\mathbf{r}_1)\ F_{q_2}(\mathbf{r}_2) \ldots F_{q_P}(\mathbf{r}_P). \quad (11)$$

In this expression, the sum contains all possible values 0, ± 1 of q_1, q_2, etc., with the only condition that their sum be zero. Now, since the integral:

$$\int_0^{2\pi} \frac{d\Phi}{2\pi}\ e^{iq\Phi} \quad (12)$$

(where q is a positive or negative integer) vanishes except if $q = 0$, we can release this condition by rewriting (11) as:

$$\sim \int_0^{2\pi} \frac{d\Phi}{2\pi} \prod_{i=1}^{P} \left\{ F_0(\mathbf{r}_i) + e^{i\Phi} F_{+1}(\mathbf{r}_i) + e^{-i\Phi} F_{-1}(\mathbf{r}_i) \right\}. \quad (13)$$

Finally, inserting (9) into this result provides the probability in the form:

$$\sim \int_0^{2\pi} \frac{d\Phi}{2\pi} \prod_{i=1}^{P} \left\{ 1 + x\ \cos\left[(\mathbf{k}_a - \mathbf{k}_b)\cdot\mathbf{r}_i + \Phi \right] \right\}. \quad (14)$$

This result is simple to interpret: the $\mathbf{r}_i$ dependence of any sequence of events is exactly the same as for a system with a well defined, but completely unknown, initial phase Φ. For each value of this phase, the $\mathbf{r}_i$ dependence of the probability factorizes and the detection events are independent; it gives exactly the usual interference pattern between two waves with wave vectors $\mathbf{k}_a$ and $\mathbf{k}_b$. But the summation of Φ between 0 and 2π destroys the factorization and correlates the events.

Let us assume for the sake of simplicity that the contrast ratio x is equal to 1. When the first detection localizes a particle at point $\mathbf{R}_1$, some information on Φ is immediately obtained, if only because the value $\Phi = (\mathbf{k}_a - \mathbf{k}_b)\cdot\mathbf{R}_1 + \pi$ becomes incompatible with the first detection. After this measurement, the probability of a second detection at point $\mathbf{r}_2$ is proportional to:

$$\int_0^{2\pi} \frac{d\Phi}{2\pi} \left(1 + \cos\left[(\mathbf{k}_a - \mathbf{k}_b)\cdot\mathbf{R}_1 + \Phi\right]\right) \times \left\{1 + \cos\left[(\mathbf{k}_a - \mathbf{k}_b)\cdot\mathbf{r}_2 + \Phi\right]\right\} \quad (15)$$

where the distribution of the values of Φ is no longer uniform: it is now given by the $\mathbf{R}_1$ dependent sinusoidal function between the first brackets inside the integral. If two measurements are initially performed with localizations at points $\mathbf{R}_1$ and $\mathbf{R}_2$, the Φ distribution is then given by the product of two sinusoidal functions, which is more peaked than the previous distribution, etc. More and more precise information on the value of Φ is progressively accumulated while more and more measurements are performed. At some point, the phase is practically determined and the detection events become quasi independent. As pointed out by Mølmer [19], this is because the two condensates tend to fuse into a single condensate under the effect of successive interference measurements. See also reference [20] for discussion of the evolution of the system towards a coherent state, which requires a number of detections comparable to the total number of atoms. For more details about the evolution of the Φ distribution function, see for instance [7,8,11].

It is interesting to note that the phase Φ plays a role which appears, mathematically, very similar to that of the so called "additional variables" (or "hidden variables") sometimes introduced to interpret the results of quantum mechanics — see for instance [21,22] for a review. If standard quantum mechanics is completed with an additional phase variable, one can consider that the measurement process *reveals* a pre-existing value of Φ, instead of *creating* the relative phase between the two condensates as in the standard interpretation of quantum mechanics. In our calculation, the motivation for introducing Φ was not a fundamental re-interpretation of quantum mechanics; it was just a convenient way to sum the many terms in (8) while maintaining the sum rule over the series of values of $\mathbf{k}$ and $\mathbf{k}'$. The precise role of the Φ integral is to ensure that the successive creation and annihilation operators bring back the system to the same initial Fock state, in other words to enforce a number conservation rule. Each term in the sum can be seen as arising from one possible

path over intermediates states in which the values of the occupation numbers vary in a given way before coming bask to their initial values. It is interesting to remark that the interference of all these paths should result in a simple phase integral, leading naturally to the introduction of an additional variable Φ. We recover the usual relation between conjugate variables, where a sum over one variable with equal weights ensures a strict conservation of the other; the phase appears here as related to fluctuations of the number of particles in intermediate states (instead of in the initial state as usual).

1.2 Three highly populated states

We now generalize the preceding discussion to three states by assuming that the initial state of the system is given by:

$$|\Phi_0\rangle = |N_a : \mathbf{k}_a;\ N_b : \mathbf{k}_b;\ N_c : \mathbf{k}_c\rangle \tag{16}$$

and evaluate the probability of particle position measurements at different points $\mathbf{r}_1$, $\mathbf{r}_2$, etc. by using (8) again. The calculation remains very similar to that of the preceding section. The only difference is that the $\mathbf{k}$ and $\mathbf{k}^{'}$ can now take three values, $\mathbf{k}_a$, $\mathbf{k}_b$ and $\mathbf{k}_c$, so that in the second line of (8) it is no longer sufficient to ensure that N_a returns to the same value; the value of N_b (or N_c) must also be controlled. This can be obtained by introducing two relative phases, Φ and $\Phi^{'}$. Another difference is that the sums in the first line of (8) now include three terms that contain the same $\mathbf{k}$'s, as well as six others that contain different $\mathbf{k}$'s and introduce cosines. Therefore, if we define:

$$N = N_a + N_b + N_c;\ x_{ab}^{'} = \frac{3\sqrt{N_a N_b}}{N} \tag{17}$$

we obtain the quantity:

$$N\left\{1 + \frac{2x_{ab}}{3}\cos\left[(\mathbf{k}_a - \mathbf{k}_b)\cdot\mathbf{r}_i + \Phi\right] + \frac{2x_{bc}}{3}\cos\left[(\mathbf{k}_b - \mathbf{k}_c)\cdot\mathbf{r}_i + \Phi^{'}\right] + \frac{2x_{ca}}{3}\cos\left[(\mathbf{k}_c - \mathbf{k}_a)\cdot\mathbf{r}_i - \Phi - \Phi^{'}\right]\right\} \tag{18}$$

and the probability is proportional to:

$$\sim \int_0^{2\pi}\frac{d\Phi}{2\pi}\int_0^{2\pi}\frac{d\Phi^{'}}{2\pi}\prod_{i=1}^{P}\left\{\frac{3}{2} + \cos\left[(\mathbf{k}_a - \mathbf{k}_b)\cdot\mathbf{r}_i + \Phi\right] + \cos\left[(\mathbf{k}_b - \mathbf{k}_c)\cdot\mathbf{r}_i + \Phi^{'}\right] + \cos\left[(\mathbf{k}_c - \mathbf{k}_a)\cdot\mathbf{r}_i + -\Phi - \Phi^{'}\right]\right\}. \tag{19}$$

The situation is therefore a direct generalization of that studied in the preceding section. Initially, the two phases Φ and $\Phi^{'}$ are completely undetermined but, as more and more position measurements are recorded, the relative phases become better and better determined. We can generalize to K systems in highly populated Fock states, for which position measurements progressively determine the value of $K-1$ phase differences. The phases have a property of transitivity [23,24]: measuring the $a-b$ phase difference and the $b-c$ phase difference provides by sum the knowledge of the $a-c$ phase difference.

Note however that, in all our calculations, we have ignored the intrinsic evolution of the system; as mentioned above, we have assumed that all measurements are made in a time that is sufficiently short to justify this approximation. Over longer periods of time, the inherent evolution of the condensates takes place and introduces phase dynamics. Any term in the Hamiltonian containing occupation numbers will tend to compete with the definition of phase introduced by the measurements [15,25]; for instance, the mean-field interactions cause a quantum spreading of the phase [26]. These effects are not included in the present article.

2 Intrinsic and induced phase

Until now, we have limited our study to initial states of the system that are products of Fock states, where the initial occupation numbers are perfectly well defined. We will now allow these numbers to fluctuate, beginning with the simple case where their difference fluctuates but their sum remains constant.

2.1 Constant total number of particles

We now assume that:

$$|\Phi_0\rangle = \sum_{Q=0}^{Q_{\max}} x_Q\ |N_a + Q : \mathbf{k}_a;\ N_b - Q : \mathbf{k}_b\rangle \tag{20}$$

where the difference between the population numbers varies between $N_a - N_b$ and $N_a - N_b + 2Q.$; we will assume that $Q \ll N_a, N_b$. The calculation remains similar to that of Section 1, the main difference being that a double sum:

$$\sum_{Q,Q^{'}} x_Q^* x_{Q^{'}} \tag{21}$$

is added in the calculation of the probability. In each term of this sum, instead of coming back to the same value of the difference $N_a - N_b$ under the effect of the sequence of a and $a\dagger$ operators, a change $Q - Q^{'}$ has to be accumulated in order to obtain a non vanishing contribution. Equation (11) is therefore now replaced by:

$$\sum_{Q,Q^{'}} x_Q^*\ x_{Q^{'}} \sum_{\{\sum q_i = Q - Q^{'}\}} F_{q_1}(\mathbf{r}_1)\ F_{q_2}(\mathbf{r}_2)\ldots F_{q_P}(\mathbf{r}_P) \tag{22}$$

or:

$$\sum_{Q,Q'} x_Q^* \; x_{Q'} \; \int \frac{d\Phi}{2\pi} e^{i(Q-Q')\Phi} \times \prod_{i=1}^{P} \left\{ F_0(\mathbf{r}_i) + e^{i\Phi} F_{+1}(\mathbf{r}_i) + e^{-i\Phi} F_{-1}(\mathbf{r}_i) \right\}. \quad (23)$$

This naturally leads to the introduction of the real positive function:

$$G(\Phi) = c \left| \sum_Q x_Q \; e^{-iQ\Phi} \right|^2 \quad (24)$$

where c is a normalization factor ensuring that the integral of $G(\Phi)$ between 0 and 2π is 1. We can now write the probability as:

$$\sim \int_0^{2\pi} \frac{d\Phi}{2\pi} G(\Phi) \prod_{i=1}^{P} \; \{1 + x \; \cos\left[(\mathbf{k}_a - \mathbf{k}_b) \cdot \mathbf{r}_i + \Phi\right]\} \quad (25)$$

which is straightforward to interpret: instead of being completely unknown between 0 and 2π, the initial phase Φ before the first measurements now has a distribution function $G(\Phi)$. This function is given by a Fourier transform of the probability amplitudes associated with the initial variations of the difference $N_a - N_b$ contained in (20). When successive position measurements begin, they all contribute to a further determination of the phase Φ, as in the simple case already discussed in Section 1. The information on the phase is therefore a combination of the information contained in the initial state and of that created by the successive position measurements.

A classical example is a "coherent state" obtained when the coefficients x_Q are given by:

$$x_Q = e^{-|\alpha|^2/2} \; \frac{\alpha^Q}{\sqrt{q!}} \quad (26)$$

where α is a parameter, which can be written:

$$\alpha = |\alpha| \; e^{i\Theta}. \quad (27)$$

The initial Φ distribution is then given by:

$$G(\Phi) \sim \left| \sum_Q \frac{|\alpha|^Q}{\sqrt{q!}} \; e^{iQ(\Phi-\Theta)} \right|^2 \quad (28)$$

which is a narrow distribution around $\Phi = \Theta$ when $|\alpha| \gg 1$; in this case, the initial value of Φ is already very well defined, so that not much additional information can be gained in successive interference measurements.

2.2 General state

In the most general case, both the sum and difference of N_a and N_b vary in the initial state:

$$|\Phi_0\rangle = \sum_N \sum_Q x_{N_a,\; N_b} |N_a : \mathbf{k}_a; \; N_b : \mathbf{k}_b\rangle \quad (29)$$

where, under the sum, N_a and N_b are given by:

$$N_a = \frac{N}{2} + Q; \quad N_b = \frac{N}{2} - Q. \quad (30)$$

In the calculation of the probability, the variables N and Q play a different role, because the sequence of a and $a^\dagger$ operators acts on Q but not on N. As a consequence, a double sum over Q and Q' still appears, as in the preceding section, but only a single sum over N. For each value of N, the calculation is the same as above, and leads to the definition of a N dependent distribution function for Φ:

$$G_N(\Phi) = \left| \sum_Q x_{(N/2)+Q,\; (N/2)-Q} \; e^{-iQ\Phi} \right|^2. \quad (31)$$

Then the sum over N can be made in a second step; one can easily see that equation (25) remains valid provided one introduces the following definition of $G(\Phi)$:

$$G(\Phi) = \sum_N G_N(\Phi). \quad (32)$$

We therefore find a situation where each value of N contributes independently to the distribution of the phase Φ. Different situations are possible. If for instance all functions $G_N(\Phi)$ are peaked around the same value of the phase, we obtain a state where the initial phase is well defined, as for the coherent state (26). But if, on the contrary, they are either non-peaked functions, or functions peaked around different values, the initial phase is uncertain; it only becomes better and better known as more and more position measurements are performed, as in Section 1.1. This shows that the spontaneous appearance of relative phase that we have discussed is not restricted to Fock states; it is actually a general property of all states involving high population numbers, provided the number of measurements remains much smaller than these populations.

3 Spin states; quantum non-locality

We now apply our calculation to particles with two internal states, which we note α and β. We consider that these states are spin 1/2 states; the notion of pseudo spin allows us to do so without loss of generality.

3.1 Calculation of the probability

The initial state of the system is assumed to be:

$$|\Phi_0\rangle = |N_a : \mathbf{k}_a, \alpha; \; N_b : \mathbf{k}_b, \beta\rangle. \quad (33)$$

We note $\Psi_\mu(\mathbf{r})$, with $\mu = \alpha, \beta$, the field operators associated with internal states α, β. The $\mathbf{r}$ dependent local density is then:

$$n(\mathbf{r}) = \Psi_\beta^\dagger(\mathbf{r}) \Psi_\beta(\mathbf{r}) + \Psi_\alpha^\dagger(\mathbf{r}) \Psi_\alpha(\mathbf{r}) \quad (34)$$

and the three components of the local spin density are:

$$\sigma_z(\mathbf{r}) = \Psi^\dagger_\beta(\mathbf{r})\Psi_\beta(\mathbf{r}) - \Psi^\dagger_\alpha(\mathbf{r})\Psi_\alpha(\mathbf{r})$$
$$\sigma_x(\mathbf{r}) = \Psi^\dagger_\beta(\mathbf{r})\Psi_\alpha(\mathbf{r}) + \Psi^\dagger_\alpha(\mathbf{r})\Psi_\beta(\mathbf{r})$$
$$\sigma_y(\mathbf{r}) = i\left[\Psi^\dagger_\beta(\mathbf{r})\Psi_\alpha(\mathbf{r}) - \Psi^\dagger_\alpha(\mathbf{r})\Psi_\beta(\mathbf{r})\right]. \quad (35)$$

The spin component in the direction of plane xOy making an angle θ with Ox is:

$$\sigma_\theta(\mathbf{r}) = e^{i\theta}\Psi^\dagger_\beta(\mathbf{r})\Psi_\alpha(\mathbf{r}) + e^{-i\theta}\Psi^\dagger_\alpha(\mathbf{r})\Psi_\beta(\mathbf{r}). \quad (36)$$

This operator, in spin space, has two eigenvalues $\eta = \pm 1$; the corresponding projectors are:

$$P_\eta(\mathbf{r},\theta) = \frac{1}{2}\left[n(\mathbf{r}) + \eta\sigma_\theta(\mathbf{r})\right]. \quad (37)$$

We now consider a measurement of one spin at point $\mathbf{r}_1$ in direction θ_1, another at point $\mathbf{r}_2$ in direction θ_2, etc.; this corresponds to the sequence of operators:

$$\sigma_{\theta_1}(\mathbf{r}_1);\ \sigma_{\theta_2}(\mathbf{r}_2);\ \sigma_{\theta_3}(\mathbf{r}_3)\ldots\ \sigma_{\theta_P}(\mathbf{r}_P). \quad (38)$$

As in Section 1, we assume assume that all $\mathbf{r}$'s are different so that all these operators commute; in addition, as already mentioned, in all this article we assume that the sequence of measurements is sufficiently brief to ignore any intrinsic evolution of the condensates. Using again the expansion of the field operators on the annihilation operators:

$$\Psi_\mu(\mathbf{r}) \ \sim\ \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\ a_{\mathbf{k};\ \mu} \quad (39)$$

we obtain the probability of a sequence of results:

$$\eta_1 = \pm 1;\ \eta_2 = \pm 1;\ldots\ \eta_P = \pm 1 \quad (40)$$

in the form of an average value in state $|\Phi_0\rangle$:

$$\langle\Phi_0|P_{\eta_1}(\mathbf{r}_1,\theta_1)\times P_{\eta_2}(\mathbf{r}_2,\theta_2)\times\ldots\times P_{\eta_P}(\mathbf{r}_P,\theta_P)|\Phi_0\rangle$$
$$\sim \sum_{\mathbf{k}_1}\sum_{\mathbf{k}'_1} e^{(i\mathbf{k}_1-\mathbf{k}'_1)\cdot\mathbf{r}_1}\sum_{\mathbf{k}_2}\ \sum_{\mathbf{k}'_2} e^{(i\mathbf{k}_2-\mathbf{k}'_2)\cdot\mathbf{r}_2}\times\ldots$$
$$\times\langle\Phi_0|\prod_i\left[a^\dagger_{\mathbf{k}'_1;\beta}a_{\mathbf{k}_1;\beta} + a^\dagger_{\mathbf{k}'_1;\alpha}a_{\mathbf{k}_1;\alpha}\right.$$
$$\left.+\eta_i\left(\ e^{i\theta_i}\ a^\dagger_{\mathbf{k}'_1;\beta}a_{\mathbf{k}_1;\alpha} + \ e^{-i\theta_i}\ a^\dagger_{\mathbf{k}'_1;\alpha}a_{\mathbf{k}_1;\beta}\right)\right]|\Phi_0\rangle. \quad (41)$$

The rest of the calculation is now very similar to that of Section 1.1. In each term contained in the product of the second line of (41), we may assume that the a and $a^\dagger$ are ordered in the "normal" order (with all the a's operators to the right, all $a^\dagger$'s to the left). This is because, in the first line of (41), when the projectors are replaced by their expressions in function of the field operators (and Hermitian conjugate), one can move all Ψ's to the right, all $\Psi^\dagger$'s to the left (since operators at different points of space commute). Now we see that, each time a $\mathbf{k}_i$ or a $\mathbf{k}'_i$ appears associated with the spin index α (or β), it is necessarily equal to $\mathbf{k}_a$ (or $\mathbf{k}_b$) to give an non-zero contribution; the operators a and $a^\dagger$ introduce numbers $\sqrt{N_{a,b}\pm n}$ as before, the only condition being that the creation and annihilation operators should be balanced in each sequence. If we assume as above that the number of measurements P is much smaller than N, the second line becomes proportional to:

$$\sim\int_0^{2\pi}\frac{d\Phi}{2\pi}\prod_{i=1}^{P}\left[1+\frac{x}{2}\ \eta_i\ \ e^{i[(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}+\theta_i]}\ e^{-i\Phi}+c.c.)\right] \quad (42)$$

where the contrast ratio x is defined by equation (10). The probability of the sequence of results η_1, η_2, ... η_P obtained at points $\mathbf{r}_1$, $\mathbf{r}_2$, ... $\mathbf{r}_P$ is then:

$$\sim\int_0^{2\pi}\frac{d\Phi}{2\pi}\prod_{i=1}^{P}\left\{1+x\ \eta_i\cos\left[(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}_i+\theta_i-\Phi\right]\right\} \quad (43)$$

which is the general result.

From now on in this section, we assume that $N_a = N_b$, so that the contrast ratio x is equal to 1. For a sequence of $+1$ results, (43) reduces to:

$$\sim\int_0^{2\pi}\frac{d\Phi}{2\pi}\prod_{i=1}^{P}\cos^2\left[\frac{(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r_i}+\theta_i-\Phi}{2}\right]. \quad (44)$$

A simpler case occurs when $\mathbf{k}_a = \mathbf{k}_b$. The probability of a sequence of P_+ results equal to $+1$ and P_- results equal to -1 then becomes:

$$\sim\int_0^{2\pi}\frac{d\Phi}{2\pi}\prod_1^{P_+}\cos^2\left(\frac{\theta_i-\Phi}{2}\right)\prod_1^{P_-}\sin^2\left(\frac{\theta_i-\Phi}{2}\right) \quad (45)$$

or, if all angles θ are equal:

$$\sim\int_0^{2\pi}\frac{d\Phi}{2\pi}\ \left[\cos\frac{\Phi}{2}\right]^{2P_+}\ \left[\sin\frac{\Phi}{2}\right]^{2P_-} \quad (46)$$

(Wallis integral).

If we compare with the situation studied in Section 1, we now have an additional element: the adjustable parameter θ_i (direction of the measured spin component in the xOy plane). This combines with the $\mathbf{r}_i$ dependent interference effect to determine the probability of each sequence of results, as shown by equation (43). When $\mathbf{k}_a = \mathbf{k}_b$, these spatial interference effects disappear, and the probability of the succession of results takes the simpler form (45), where the effect of the η's and θ's is isolated. For the first measurement, the two results $\eta = \pm 1$ are equally probable but, as soon as one of these results is obtained, the distribution of Φ is changed. The basic process behind the progressive determination of Φ is the same as in Section 1: after a given sequence of measurements with known results, the new distribution of probability for Φ is given (within a normalization constant) by the initial probability of this sequence, i.e. by replacing in (43) the measured η_i's by the corresponding results. Each result $\eta_i = +1$ brings in

a factor $\cos^2(\theta_i - \Phi)/2$ that tends to localize the Φ distribution around $\Phi = \theta_i$ and makes it vanish in the opposite direction $\Phi = \theta_i + \pi$; each result $\eta_i = -1$ brings in a factor $\sin^2(\theta_i - \Phi)/2$ that does exactly the opposite. A combination of two successive opposite results $\eta = +1, -1$ brings in a factor $\sim\sin^2(\theta_i - \Phi)$ that cancels the Φ distribution in the two directions $\Phi = \theta_i$ and $\Phi = \theta_i + \pi$ and tends to localize it around $\Phi = \theta_i \pm \pi/2$. Similarly, sequences of η_is with different numbers of results $+1, -1$ will introduce a Φ maximum in some intermediate direction (see for instance the discussion of Sect. III-C of Ref. [8]).

This illustrates how a sequence of successive measurements can progressively make the Φ distribution more and more peaked around some value $\Phi = \Phi_{\max}$, and brings the system closer to a situation where all the spins are polarized in the same transverse direction $\theta = \Phi_{\max}$; interesting numerical simulations of the phenomenon can be found in reference [27]. The individual discrete measurements cooperate to provide information on a quasi-continuous quantity, the direction of the transverse orientation. Moreover, the experimenter can adjust the choice of the measurement angles θ_i in function of the preceding results ± 1 obtained. It is not necessary to orient the measurement apparatus in the direction of the component that is to be measured, as would be the case for a single spin; a possible strategy to measure the position of the maximum $\Phi_{\max}$ is to perform measurements in the perpendicular direction and check that the proportion of $\eta \pm 1$ results is $1/2$ (in this way, first order θ variations can be measured).

3.2 Quantum amplification

An interesting property of the spin measurement sequence is that it can give information on the transverse spin orientation of a very large number $2N$ of particles even when P, the number of actually measured particles, is much lower. In principle, one can get information on the spin orientation of a macroscopic sample by just measuring the spin orientation of a microscopic sub-sample made of 100 particles for instance, which involves a huge amplification factor. Such an amplification is often evoked in the theoretical discussion of the measurement process in quantum mechanics, where a microscopic system (the measured system) triggers an instability in the measurement apparatus, leading to macroscopically different states of the pointer (here the large number of non-measured particles). The phenomenon is clearly related to the effect discussed by Siggia and Ruckenstein [28] in the context of the spontaneous appearance of a transverse spin orientation in a double condensates of spin polarized atomic hydrogen.

A related discussion is given by Leggett and Sols in reference [15] (see paragraph before last of this reference), who consider two superconducting systems that are initially described by an incoherent mixture of number states, and between which a Josephson current can flow. They then ask the question "does the act of looking to see whether a current flows itself force the system into an eigenstate of current and hence of relative phase?". They assume the presence of a small compass needle measuring the magnetic field produced by the Josephson current. Both the needle and the current are macroscopic, but one can assume that the current is arbitrarily large while the needle is tiny. Under these conditions, it seems "bizarre in the extreme" to assume that, by some mysterious amplification effect, it is the small classical object that will force the large one to take a definite value. The authors conclude that, while standard mechanics would answer "yes" to their question, "common sense rebels against this conclusion, ... and we believe that in this case common sense is right" — in other words they conclude that the phase Φ existed before the measurement (additional variable). In our case, the discussion is different since no external classical pointer (the compass needle) is introduced. In a sense, the point is even stronger since the small object forcing the large classical object into a definite value may even be microscopic. But the conclusion remains the same: if one rejects the idea of a small object creating the value of a macroscopic variable (the phase), then one is led naturally to accept the existence of hidden variables in quantum theory. We note in passing that, as remarked by Bell [29,30], even if it is traditional, it is somewhat clumsy to call "hidden" an additional variable such as Φ: it is precisely the variable that is actually seen in the experiment, while the variables of quantum mechanics (wave functions) tend to remain invisible.

3.3 Non-local quantum effects

For the discussion of this subsection, it is convenient to assume that the orbital states of the particles are not necessarily plane waves. We therefore replace (33) by:

$$|\Phi_0\rangle = |N_a : \varphi_a, \alpha;\ N_b : \varphi_b, \beta\rangle \tag{47}$$

where the orbital states correspond to the wave functions:

$$\langle \mathbf{r}|\varphi_a\rangle = \varphi_a(\mathbf{r}), \quad \langle \mathbf{r}|\varphi_b\rangle = \varphi_b(\mathbf{r}) \tag{48}$$

with relative phase given by:

$$\arg\{\varphi_a(\mathbf{r})/\varphi_b(\mathbf{r})\} = \xi(\mathbf{r}). \tag{49}$$

The calculation remains similar to that given above. In (39), the field operator was expanded on the annihilation operators $a_{\mathbf{k}}$ corresponding to plane waves; here we expand $\Psi_\alpha(\mathbf{r})$ on annihilation operators corresponding to a base where the particle is in internal state α and in orbital states with wave functions $\varphi_a(\mathbf{r})$, $\varphi_2(\mathbf{r})$, $\varphi_3(\mathbf{r})$, etc.:

$$\Psi_\mu(\mathbf{r}) \ \sim \ \sum_{q=\mathbf{1}}^{\infty} \varphi_q(\mathbf{r})\ a_{q;\ \mu} \tag{50}$$

with $\varphi_1(\mathbf{r}) = \varphi_a(\mathbf{r})$, and where $\varphi_2(\mathbf{r})$, $\varphi_3(\mathbf{r})$, etc. complete $\varphi_a(\mathbf{r})$ in order to make an orthonormal basis in orbital space. We do something similar for $\Psi_\beta(\mathbf{r})$, now in a basis with orbital states that contain $\varphi_b(\mathbf{r})$ as the first vector. In practice, the only difference is that the exponentials

$e^{i[(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}_i]}$ are now replaced by the product $\varphi_b^*(\mathbf{r})\varphi_a(\mathbf{r})$. The result for the probability is then:

$$\sim \int_0^{2\pi} \frac{d\Phi}{2\pi} \prod_{i=1}^{P} \Big[|\varphi_a(\mathbf{r}_i)|^2 + |\varphi_b(\mathbf{r}_i)|^2 + 2x\eta_i \, |\varphi_a(\mathbf{r}_i)\varphi_b(\mathbf{r}_i)| \;\; \cos\left[\xi(\mathbf{r}_i) - \theta_i + \Phi\right] \Big]. \quad (51)$$

Even if $N_a = N_b$ (so that $x = 1$) the contrast ratio of the interference term, which contains the θ_i dependence of the result, is not necessarily 1; it is maximum at points $\mathbf{r}$ where they have the same modulus, zero if one of them vanishes. Clearly, in order to determine the relative phase Φ, transverse spin measurements should be made in the regions of good overlap between $\varphi_a(\mathbf{r})$ and $\varphi_b(\mathbf{r})$.

We can now assume that $\varphi_a(\mathbf{r})$ and $\varphi_b(\mathbf{r})$ are equal and that both these functions are non-zero only in two distant regions of space, D and $D^{'}$; for simplicity we will assume that they are constant within these two domains; equation (51) then simplifies and the orbital wave functions disappear, as in (45). All the measurements are made in region D, which is supposed to be relatively small so that it contains a small average number of particles, 10 or 100 for instance. Region $D^{'}$ can be much larger and contain a macroscopic number of particles. Both regions can be very distant from each other, so that no signal transmission between them is possible during the duration of the experiment. The curious prediction of quantum mechanics is that a measurement of a microscopic transverse spin orientation in region D will immediately induce the appearance of a parallel macroscopic spin orientation in region $D^{'}$.

One can make the situation even more striking by assuming that the wave functions $\varphi_{a,b}(\mathbf{r})$ are non-zero in three regions of space: a small measurement region D as before, a large region $D^{'}$ at a small distance from D, and finally another large region $D^{''}$ very far away (in a different galaxy for instance). The transverse spin orientation in $D^{'}$ then plays the role of a local pointer, which can be seen as a part of the measurement apparatus which measures the direction of the transverse spin orientation in D. Remote region $D^{''}$ contains another pointer, which under the effect of a measurement in D immediately takes a direction parallel to that of the pointer in $D^{'}$; the phenomenon could then be called teleportation of the direction of pointers, a clear illustration of quantum non-locality. Of course, we should also make the usual proviso: the direction obtained in the measurement can not be controlled, but is completely random: there is no way to use the teleportation to transmit information [31,32], a necessary condition for preserving consistency with relativity.

3.4 The EPR argument for condensates

The perfect correlation between the direction of pointers at arbitrary distances is of course reminiscent of the quantum correlations discussed in the context of the famous Einstein-Podolsky-Rosen (EPR) argument; the similarity is even more striking in the EPRB version proposed by Bohm (B), which involves the measurement of spin directions. For a review, see for instance reference [22] and references therein; a detailed historical perspective is given in [33]. EPRB consider two correlated particles which undergo spin measurements in two remote regions of space and assume that quantum mechanics gives correct predictions concerning the probabilities of the various measurements of the spins along different directions. The EPR reasoning [34] states that "... if, without in any way disturbing a system we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity" (may the most quoted sentence of all physics!). Since EPR attribute independent elements of physical reality to the content of remote regions of space, this is often called an assumption of "local realism". From this they show that some elements of physical reality are not contained in quantum mechanics, in other words that this theory is incomplete. Bohr rejected this conclusion [35] because he considered that the notion of element of reality used by EPR contained essential ambiguities. In his view, "the procedure of measurement has an essential influence on which the very definition of the physical quantities in question rests". One should then only ascribe physical reality to the whole extended system made of the microscopic particles *and* the macroscopic measurement apparatus, not to microscopic subsystems only. As a consequence, one can not always attribute distinct physical properties to the content of in different regions of space, as EPR do. In other words, Bohr sees the process of measurement on an extended microscopic system as a fundamentally random process that is not necessarily localized in space [36].

Let us now apply the EPR reasoning to the evolution of the macroscopic pointers (transverse spin orientation) in $D^{'}$ and $D^{''}$, when a series of spin measurements is performed in D. We focus for a moment on $D^{''}$ and consider the elements of physical reality associated with this region of space. As EPR, we consider that they can not vary suddenly under the effect of events taking place at arbitrarily large distances in D and $D^{'}$. Now compare the elements of reality associated to region $D^{''}$, just before, and just after the series of spin measurements performed in region D. After the measurement the region contains a macroscopic spin orientation in a given transverse direction, which are necessarily associated with some elements of reality. But these elements cannot appear as a result of a random process taking place in D, with no possible causal link since the distance is much too large. Therefore, even before the measurement, the element(s) of reality associated with the spin orientation already existed. But, on the other hand, standard quantum mechanics does not include this element before the measurement, only after; it therefore misses at least one element of reality, it is incomplete.

One can make the reasoning more explicit by considering two realizations of the experiment, one in which the transverse spin orientation is found at the end of the

experiment in a direction $\Phi = \Phi_1$ (with some small uncertainty $\Delta\Phi$ depending on the number or measurements), another in which this spin orientation is found in a different direction $\Phi = \Phi_2$, after exactly the same sequence of measurements. Standard quantum mechanics describes the two final situations by different vectors in the space of states of the system, accounting for the existence of different physical quantities in region D'' (different elements of reality). On the other hand, it considers that the initial states were exactly the same; the differences appear only under the effect of the measurements. But this is in opposition with the EPR notion of local reality, which implies that the differences must have already existed before the measurements. In other words, the EPR reasoning shows that the two realizations of the experiment actually started from different initial conditions. Since quantum mechanics ignores this difference, it is incomplete.

What are the consequences of this transposition of the EPR reasoning to a different situation than the two particles of the original article? First, the EPR sentence quoted in the first paragraph of this subsection becomes even more true. Here the pointers in D' and D'' are not invisible microscopic objects, but macroscopic and permanent angular momenta, similar to the spin magnetization of ferromagnets; they are classical physical entities that can be manipulated directly, so that it seems difficult to deprive them of properties that are independent of the measurement apparatuses. Therefore Bohr's denial of independent physical reality becomes more difficult to accept (of course, no one knows if Bohr would have given again it in this context!). Second, even if one follows Bohr and takes for granted that physical properties are reserved to systems including macroscopic measurement apparatuses, another difference arises here: one single experimental setup can lead to many spin orientations. With a single spin, one can measure only the spin component along the direction defined by the apparatus; with many spins in region D, it is the measurement process which determines the axis along which the spins become oriented (the transverse orientation can take in general a direction which makes any angle with the direction of measurement). In order to show that incompatible observables can simultaneously be defined (as opposed to standard quantum mechanics), the usual EPR argument involves different (and incompatible) experimental setups. Here, no measurement at all is performed in D' and D''; moreover, the sequence of measurements performed in D may be always exactly the same in successive realizations of the experiment, even if they lead to different directions of the transverse orientation[1]. Clearly, Bohr's argument ascribing different physical realities to incompatible measurements does not apply when only one kind of measurement takes place. Another difference is that, in the usual discussion of EPR correlations, the entanglement of the two spin particles plays an essential role; here, the initial state vector is a simple product involving the two internal spin states. One can finally remark that, in usual EPRB situations, one has to invoke a "no quantum cloning" theorem to explain why superluminal signaling is impossible. Here, in all regions D, D' and D'', we have a large number of copies of the individual systems to measure precisely the transverse direction of polarization.

The EPR argument therefore certainly looks different, and probably even stronger, in the context of Bose-Einstein spin condensates. Within standard quantum mechanics, it seems really difficult not to speak of action at a distance to describe the instantaneous appearance of the macroscopic transverse orientation of the pointer in region D'' — but, again, we must remember that its final direction is totally uncontrolled and cannot be used as a signal. In other words, what we we have is an "uncontrolled action at a distance". On the other hand, we do not have the equivalent of a Bell theorem in this case: the introduction of an additional variable Φ to quantum mechanics easily allows one the reproduce the prediction of quantum mechanics in a purely local model. Amusingly, the situation is exactly the opposite of that usually discussed in the context of EPR-Bell experiments on pairs of spins: here non-locality appears clearly in standard quantum mechanics, but disappears in theories with additional variables.

3.5 Conservation of angular momentum

An intriguing question relates to the momentum absorbed by the measurement apparatus during the interaction with the measured system. For instance, in a usual Stern-Gerlach experiment where a single spin, initially polarized in a transverse direction, is measured to end up in a longitudinal direction (with respect to the measurement apparatus), it is generally assumed that the variation of its angular momentum is absorbed by the apparatus — this variation is microscopic and so small that it is impossible to measure in practice. Here, we have a system which starts before measurement from a state (47) with zero average value of the angular momentum[2]; after a sequence of measurements performed in D it reaches another state where all spins point in a transverse direction given by the spontaneous value of Φ, which has a high angular momentum if there are many spins. From the usual conservation rule, one could then expect a priori that the measurement apparatus should acquire the opposite amount of angular momentum. On the other hand, we have seen that the

[1] If region D' contains a large average number or particles, its spin orientation is classical; within a good approximation, all its components correspond to communing operators, and they are not incompatible quantities in standard quantum mechanics. On the other hand, if regions D' contains only one or two particles on average, orthogonal components of the spin orientation are no longer compatible observables, allowing one to transpose the usual EPR argument on incompatible experiments to this case.

[2] We reason within standard quantum mechanics, assuming that it is the measurement process that puts the system into a state with definite angular momentum; we do not assume that the process only reveals a value that existed before, as in theories with additional variables.

measurement of a few spins in D may result in the appearance of a macroscopic angular momentum for the whole system in $D + D'$. Does the measurement apparatus really absorbs all the angular momentum acquired by the whole system, although it interacts directly only with a subsystem in D that is very small?

The usual rules of quantum mechanics are not very specific about the answer to this question. The emphasis is usually put on measurements of microscopic systems, which have a completely negligible effect on the macroscopic measurement apparatus — an exception, nevertheless, is the discussion of the recoil effect of a moving double slit experiment in the famous Bohr-Einstein debate. Here, due to the amplification process discussed in Section 3.2, the measured system itself is macroscopic. If we assume that the answer to the question is "yes", in order to keep a strict conservation rule for the angular momentum, we open the way to paradoxes: for instance, by locally dephasing in region D' the two wave functions φ_a and φ_b, before the measurement in D, one would control the direction of the angular momentum in D' induced by this measurement, and therefore the transfer of angular momentum to the apparatus in region D. This would allow superluminal signaling, in contradiction with relativity. The appropriate answer is therefore probably rather "no, the angular momentum transferred is just that of the individual spins that are measured"; in this perspective, the measurement operation creates in region D' an angular momentum that appears from nothing, which is somewhat paradoxical too. Maybe this just means that the rule of angular momentum conservation should to not be taken too strictly in a quantum measurement processes. Another possibility, again, is to take the point of view of theories with additional variables, and assume that Φ was initially actually perfectly determined in each realization of the experiment, even if it is unknown; in this perspective no angular momentum transfer has to take place, so that the difficulty vanishes.

3.6 More than two internal levels

The above considerations can easily be transposed to more than two internal states, three for instance. The situation then becomes analogous to that of Section 1.2, where two relative phases Φ and Φ' enter the calculation instead of one. One can assume that transverse spin observables relative to the $\alpha - \beta$, and to the $\beta - \gamma$, transitions are measured. A sufficient number of measurements will determine the two relative phases. An interesting property is that the transverse orientation of the spin associated to the $\alpha - \gamma$ transition will then be determined, although no direct measurement of this observable has been made.

4 Conclusion

In standard quantum mechanics, physical systems in highly populated Fock states have a completely undetermined relative phase, but this phase can become very well defined under the effect of a few interference measurements only; this is true even in the number of particles detected is negligible with respect to the total populations. The results mimic exactly what would happen if the initial phase was fixed, but initially unknown, and progressively revealed by the observation of the interference pattern. So, the answer to the question mentioned in the introduction "do condensates that have never seen each other have a relative phase?" is: in standard quantum mechanics, in principle they do not, but they can acquire it under the effect of a few microscopic measurements, in a way which gives the impression of a well defined initial phase. As Anderson puts it [18]: "any future experiment can be interpreted as if Φ was fixed". It is therefore tempting to assume that the phase existed from the beginning. This is the point of view of physicists who argue that, for superfluids, the usual postulates of quantum mechanics should be completed by an additional postulate introducing spontaneous symmetry breaking (of the conservation of the number of particles); this leads to non-zero average value of the field operator. This view is of course legitimate, but is actually nothing but another form of the "hidden variable" theory of quantum mechanics (de Broglie, Bohm), as we have discussed in Section 1.1. It also remains legitimate to stick to orthodox quantum mechanics and to consider that it is the process of quantum measurement itself that acts on the condensates and forces them to acquire a well defined relative phase.

Spin adds to this problem the notion of angular momentum and offers interesting variants of the usual EPR non-locality experiments. One can suppose that spin orientation measurements are performed in two (or more) very remote regions of space. An interesting possibility is that some of the remote systems may be macroscopic; the results in terms of transverse direction of the local spin orientation of the sample are not discrete, as in usual EPR experiments, but continuous. Actually the macroscopic spin orientation plays here the role of the usual pointer often considered in the theory of quantum measurement, but with a surprising property: the pointer can be put at an arbitrary distance from the measured system. In one region of space, a series of measurements determines a transverse orientation, in another remote region a classical macroscopic pointer immediately takes the corresponding direction and stays there, accessible to future measurements! It seems difficult to deny that standard quantum mechanics involves action at a distance in this case. This action is instantaneous, but not controllable, so that it can not be used to send superluminal signals. There has been a long debate [37–40] to decide whether or not non-locality is an inherent property of quantum mechanics, or if it appears only if additional ingredients are added to the theory (hidden variables, local reality, counterfactuality [41], etc.). Our example can be discussed just in terms of standard quantum mechanics, without adding such additional elements and/or incompatible experimental setups, which would allow to invoke different physical realities in a Bohr type argument. It therefore clearly speaks in favor of accepting non-locality as an intrinsic property of standard quantum mechanics.

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